Indian Statistical Institute

B. Math (Hons.) second year

Computer Science II- Numerical Methods

May 2, 2018

Final exam | Time: 3 hours | 50 points

1. [5 points] In Octave/Matlab, what do `eps`, `realmin` and `realmax` represent? What is `overflow` and `underflow`?

2. [5 points] Suppose $f \in C^n[a,b]$ and $x_0, x_1, \ldots, x_n$ are distinct numbers in $[a,b]$. Then show that there exists a number $\zeta \in (a,b)$ with $f[x_0, x_1, \ldots, x_n] = \frac{f^n(\zeta)}{n!}$.

3. (a.) [2 points] Define a cubic spline interpolant of a function f on $[a,b]$.

   (b). [3 points] A natural cubic spline S on $[0,2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3 & \text{if } 0 \le x < 1 \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3 & \text{if } 1 \le x \le 2. \end{cases}$$

   Find b, c and d.

4. [5 points] Let

$$A = \begin{bmatrix} 1 & -3 & 7 \\ -2 & 6 & 1 \\ 0 & 3 & -2 \end{bmatrix}$$

   Find P, L and U such that $PA = LU$ where P is permutation matrix, L is lower triangular with $1's$ on its diagonal and U is upper triangular.

5. (a). [1 point] Define strictly diagonally dominant matrix of order n.

   (b). [3+3 points] Prove that a strictly diagonally dominant matrix A is non- singular and Gaussian elimination can be performed on any linear system of the form $Ax = b$ to obtain its unique solution without row or column interchange.

   (c). [5 points] Write an Octave/Matlab function that determines whether a matrix is strictly diagonally dominant. The input to the function should be a matrix, and the output should be one if the matrix is strictly diagonally and zero if it is not.

7. [6 points] Evaluate $I = \int_0^1 e^{-x^2} dx$ using Gauss- Legendre quadrature rule of order two, round up to 5 decimal places, for $N = 1$ and $N = 2$ where N is the number of partitions of equal size $H = \frac{1}{N}$ of $[0,1]$. The Legendre polynomial of degree 2 is $\varphi_2(x) = \frac{1}{2}(3x^2 - 1)$.

8. (a). [5 points] Suppose f is continuous and satisfies the Lipschitz condition with constant L on $D = \{(t,y) \mid a \le t \le b, -\infty < y < \infty\}$ and that a constant M exists with

$$|y''(t)| \le M, \ \forall t \in [a,b]$$

   where $y(t)$ denotes the unique solution to IVP,

$$y'(t) = f(t,y), \ a \le t \le b, \ y(a) = \alpha.$$

   Let N be a positive integer, $h = \frac{b-a}{N}$ and $t_i = a + ih, \ i = 0, 1, \ldots, N$. Let $w_0, w_1, \ldots, w_N$ be the approximations generated by Euler's method. Then for each $i = 0, 1, 2, \ldots, N$, prove that

$$|y(t_i) - w_i| \le \frac{hM}{2L}[e^{L(t_i - a)} - 1]$$

(b). [5 points] Write an Octave/Matlab code to implement Euler's method.

(c). [2 points] Show that the following IVP

$$\frac{dy}{dt} = \frac{2}{t}y + t^2e^t, \ 1 \le t \le 2, \ y(1) = 0,$$

has unique solution.